ON THE RANGE OF AN INVARIANT MEAN

BY
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Introduction. Let S be a discrete semigroup, $m(S)$ the space of bounded real functions on S with the usual sup. norm, and $m(S)^*$ the conjugate Banach space of $m(S)$. An element $\phi \in m(S)^*$ is a mean if $\phi(f) \geqq 0$ whenever $f \geqq 0$ and $\phi(1)=1$, where 1 denotes also the constant one function on S.

S is said to be left [right] amenable if there is a mean $\phi \in m(S)^*$ which is in addition left [right] invariant, i.e., satisfies $\phi(f_a)=\phi(f)$ $[\phi(f^a)=\phi(f)]$ for each $f \in m(S)$ and $a \in S$ (where $f_a(s)=f(as)$ and $f^a(s)=f(sa)$ for f in $m(S)$ and $a, s \in S$). S is amenable if there is a mean $\phi \in m(S)^*$ which is left and right invariant.

If $A \subset S$, then 1_A will be the function which is one on A and zero otherwise. We shall write 1 instead of 1_S and $\phi(A)$ instead of $\phi(1_A)$ (if $\phi \in m(S)^*$), sometimes.

The range of an element $\phi \in m(S)$ is the set of numbers $\{\phi(A)$, where A ranges over all subsets of $S\}$. It is clear that the range of a mean is a subset of $[0, 1]$ $=\{x; 0 \leqq x \leqq 1\}$.

If S is a left amenable semigroup, define the following relation between elements of S: $a \sim b$ iff $as=bs$ for some s in S. The relation $\sim$ is an equivalence relation which is two-sidedly stable (or a congruence), i.e., if $a \sim b$, then $as \sim bs$ and $sa \sim sb$ for any s in S (since S is left amenable) (see [3, p. 371] and Ljapin [1, p. 39]).

If $s \in S$, let s' stand for the equivalence class to which s belongs and let $S'=\{s'; s \in S\}$. A multiplication between the elements of S' can be defined by $s't'=(st)'$. This multiplication is well defined and associative, rendering thus S' a semigroup (since $\sim$ is a congruence. See Ljapin [1, pp. 265–266]). Furthermore, S' has right cancellation (and coincides with S if S has right cancellation).

It has been shown by this author in [4] (Corollary to Lemma 1) that, if S has right cancellation, is left amenable and contains an element of infinite order, then the range of any invariant mean on $m(S)$ contains the set of rationals in $[0, 1]$. Moreover, if $0 \leqq r \leqq 1$ is a rational number, then there is a set $A \subset S$ such that $\phi(A)=r$ for *any* left invariant mean ϕ on $m(S)$.

T. Mitchell has even suggested orally a proof to show that, for the above considered semigroup S, the range of each left invariant mean ϕ on $m(S)$ is the whole $[0, 1]$ interval.

We would like to prove in this paper the following:

CONJECTURE. Let S be a left amenable semigroup. Then the range of each left invariant mean on $m(S)$ is the whole $[0, 1]$ interval if and only if S' is infinite.

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Unfortunately, we do not know how to prove this result. But we know how to prove something close to it.

Call a group G an AB group if: (a) G is amenable, (b) each element of G has finite order, (c) each infinite subgroup of G is *not* locally finite (a group is locally finite if each finitely generated subgroup is finite), (d) G is an infinite group.

We doubt whether AB groups exist at all, since if G were an AB group, then each of its infinite subgroups would contain a subgroup which would provide a counter-example to Burnside's conjecture and would be in addition, amenable.

The main result of this paper is the

THEOREM 3. *Let S be a left amenable semigroup for which S' is not an AB group. Then the range of each left invariant mean on $m(S)$ is the whole* $[0, 1]$ *interval if and only if S' is infinite.*

It comes out that if S' is not an AB group, then $m(S)$ admits a left invariant mean whose range is not the whole $[0, 1]$ interval, if and only if S' *is a finite group*. In this case, there is even a left invariant mean on $m(S)$ (which may annihilate single point sets) whose range is the set $\{k/n;\ k=0, 1, \dots, n\}$, where n is the order of S'.

We prove in fact, more than stated in the above theorem, namely that if S' is an infinite non-AB group, then there is a class of subsets of S $\{A(x),\ 0 \leqq x \leqq 1\}$ which satisfies $A(0)=\varnothing$, $A(1)=S$, $A(x)\subset A(x')$ if $0\leqq x\leqq x'\leqq 1$ and $\phi(A(x))=x$ for *each* left invariant mean $\phi \in m(S)^*$ and each $0\leqq x\leqq 1$. In particular, the range of a left invariant mean is attained on the class of sets $A\subset S$, for which $\phi(A)$ is constant when ϕ ranges over the set of left invariant means.

In conclusion, it is a pleasure to thank T. Mitchell for the friendly and fruitful conversations we had with him. The idea of the proof of Theorem 1 and of the lemma preceding it, was basically suggested by him.

Some more notations. Let S be a semigroup. If $a \in S$, we denote by $l_a\colon m(S)\to m(S)$, defined by $l_a f=f_a$ for f in $m(S)$. If $\phi\in m(S)$, then $L_a\colon m(S)^*\to m(S)^*$ is defined by $(L_a\phi)f=\phi(f_a)$ for each f in $m(S)$. If $A\subset S$ we say that $\phi\in m(S)^*$ is A-left invariant if $L_a\phi=\phi$ for each a in A. If A consists of the single element $c\in S$, then we say that ϕ is c-left invariant. If A, B are subsets of S, then B is A-left almost convergent if $\phi_1(B)=\phi_2(B)$ for any two A-left invariant means ϕ_1, $\phi_2\in m(S)^*$. $B\subset S$ is left almost convergent if it is S-left almost convergent. For characterizations of almost convergent subsets of the additive semigroup of positive integers, see G. Lorentz [6] (see also Day [2, pp. 538–540]).

If S is a right cancellation semigroup, then $c\in S$ has order $n\geqq 1$, if $n+1$ is the first integer which satisfies $c^{n+1}=c$. $c\in S$ has infinite order if no such $n\geqq 1$ exists.

The main theorem.

LEMMA 1a. *Let S be a right cancellation semigroup, $c\in S$ an element of infinite* *order, and $A \subset S$ be such that $cA \subset A$. Let $Q_n = \{p/2^n;\ p = 0, 1, \ldots, 2^n\}$ and $Q = \bigcup_{n=1}^{\infty} Q_n$. Then there exists a collection of sets $A(r)$, $r \in Q$, such that*

(1) $A(0) = \varnothing$, $A(1) = A$.

(2) *If $r, r' \in Q$ with $r < r'$, then $A(r) \subset A(r')$.*

(3) *$\phi(A(r)) = r\phi(A)$ for any c-left invariant mean ϕ.*

Proof. By Lemma 1 of [4], there are disjoint sets V_1, $V_2 \subset A$ such that $A = V_1 \cup V_2$, $cV_1 \subset V_2$ and $cV_2 \subset V_1$ (hence $c^2V_1 \subset V_1$). Let $A(0) = \varnothing$, $A(\frac{1}{2}) = V_1$, and $A(1) = A = V_1 \cup V_2$. Then

$$c^2[A(\tfrac{1}{2}) - A(0)] \subset [A(\tfrac{1}{2}) - A(0)], \quad c^2[A(1) - A(\tfrac{1}{2})] \subset [A(1) - A(\tfrac{1}{2})],$$

and if ϕ is any c-left invariant mean, then $\phi(V_1) \leqq \phi(cV_1) \leqq \phi(V_2) \leqq \phi(cV_2) \leqq \phi(V_1)$, i.e., $\phi(V_1) = \phi(V_2)$. Hence, $\phi(A(\frac{1}{2})) = \frac{1}{2}\phi(A)$ and so $\phi(A(r)) = r\phi(A)$ for any $r \in Q_1$. For fixed n, arrange the set of rationals Q_n in increasing order as $0 = r_0 < r_1 < \cdots < r_{2^n} = 1$ and assume that we have found sets $A(r) \subset A$, $r \in Q_n$ such that $A(r) \subset A(r')$ if $r, r' \in Q_n$ with $r < r'$, $A(0) = \varnothing$, $A(1) = A$ and such that $\phi(A(r)) = r\phi(A)$ for any c-left invariant mean ϕ and any $r \in Q_n$ and such that for some $k > 0$,

$$c^k[A(r_i) - A(r_{i-1})] \subset [A(r_i) - A(r_{i-1})]$$

for $1 \leqq i \leqq 2^n$.

There are then, by Lemma 1 of [4], disjoint sets $V_1^{(i)}$, $V_2^{(i)}$ with

$$V_1^{(i)} \cup V_2^{(i)} = A(r_i) - A(r_{i-1}),$$

$c^kV_1^{(i)} \subset V_2^{(i)}$, and $c^kV_2^{(i)} \subset V_1^{(i)}$. If ϕ is any c-left invariant mean then as before $\phi(V_1^{(i)}) = \phi(V_2^{(i)})$ and so

$$2\phi(V_1^{(i)}) = \phi[A(r_i) - A(r_{i-1})] = (r_i - r_{i-1})\phi(A) = \phi(A)/2^n.$$

Hence $\phi(V_1^{(i)}) = \phi(A)/2^{n+1}$. Define now: $A(r_{i-1} + 1/2^{n+1}) = A(r_{i-1}) \cup V_1^{(i)}$ for $1 \leqq i \leqq 2^n$. Since $V_1^{(i)} \cap A(r_{i-1}) = 0$, we have that

$$\phi(A)(r_{i-1} + 1/2^{n+1}) = (r_{i-1} + 1/2^{n+1})\phi(A)$$

for any c-left invariant mean. For $r \in Q_{n+1}$ and $r \notin Q_n$ we have thus defined sets $A(r)$. If $r \in Q_n$, let $A(r)$ be given by the induction hypothesis. (Hence the sets $A(r)$ constructed at stage $n+1$ coincide for $r \in Q_n$ with the sets built at stage n.) We have, from above, that $A(r_i) - A(r_{i-1} + 1/2^{n+1}) = V_2^{(i)}$ and $A(r_{i-1} + 1/2^{n+1}) - A(r_{i-1}) = V_1^{(i)}$. Hence, if the elements of Q_{n+1} are arranged in increasing order as $0 = s_0 < s_1 < \cdots < s_{2^{n+1}}$, then $c^{2k}[A(s_i) - A(s_{i-1})] \subset A(s_i) - A(s_{i-1})$, $A(s_{i-1}) \subset A(s_i)$ for $1 \leqq i \leqq 2^{n+1}$ and $\phi(A(s_i)) = s_i\phi(A)$ for any c-left invariant mean. Consider now the sets $A(r)$, $r \in Q = \bigcup^{\infty} Q_n$. If $r, r' \in Q$ with $r < r'$, then $r, r' \in Q_n$ for some n (since $Q_n \subset Q_{n+1}$); hence $A(r) \subset A(r')$. Conditions (1), (3) are satisfied from above.

THEOREM 1. *Let S be a right cancellation semigroup, $c \in S$ an element of infinite order, and $A \subset S$ be such that $cA \subset A$. Then there exists a collection of sets $\{A(x); 0 \leqq x \leqq 1\}$ such that*:

(1) $A(0) = \varnothing$, $A(1) = A$.

(2) *If* $0 \leqq x < x' \leqq 1$, *then* $A(x) \subset A(x')$.

(3) $\phi(A(x)) = x\phi(A)$ *for any c-left invariant mean* ϕ *on* $m(S)$ *and any* $x \in [0, 1]$.

Proof. For $x \in Q = \bigcup_{n=1}^{\infty} Q_n$, let $A(x)$ be the sets constructed in the previous lemma. If $0 < x < 1$ is such that $x \notin Q$, define $A(x) = \bigcap_{\{r \in Q; r > x\}} A(r)$. If now $0 \leqq x < x' \leqq 1$, then:

(a) if $x \in Q$, then $A(x) \subset A(r)$ for any $r > x'$; hence $A(x) \subset A(x')$,

(b) if $x' \in Q$, then $A(x) \subset A(x')$ is clear, and

(c) if $x \notin Q$ and $x' \notin Q$, then for some $r \in Q$ we have $x < r < x'$ which implies $A(x) \subset A(r) \subset A(x')$. Hence, (1) and (2) hold.

If now $0 < x < 1$ and $x \notin Q$ and ϕ is a c-left invariant mean, then

$$r\phi(A) = \phi(A(r)) \leqq \phi(A(x)) \leqq \phi(A(r')) = r'\phi(A)$$

for any $r, r' \in Q$ with $r < x < r'$. This implies that $x(\phi(A)) \leqq \phi(A(x)) \leqq x\phi(A)$ since Q is dense in $[0, 1]$, which finishes this proof.

COROLLARY. *Let S be a right cancellation semigroup, $c \in S$ an element of infinite order. There exists a class of sets $\{A(x); 0 \leqq x \leqq 1\}$ such that*:

(1) $A(0) = \varnothing$ *and* $A(1) = S$.

(2) $A(x) \subset A(x')$ *if* $0 \leqq x < x' \leqq 1$.

(3) $\phi(A(x)) = x$ *for any* $0 \leqq x \leqq 1$ *and any c-left invariant mean* $\phi \in m(S)^*$.

The idea of using the background of [4] (see [4, Lemma 1 and Corollary to Lemma 1]) as above and especially the use of the monotonicity of an invariant mean as in the proof of Theorem 1 is due to T. Mitchell (oral communication).

REMARKS. Let X be a set, $\mathscr{A}$ an algebra of subsets of X and ν a finitely additive, finite measure on $\mathscr{A}$. Then ν has the intermediate property on $A \in \mathscr{A}$ if for any $0 \leqq b \leqq \nu(A)$ there exists a $B \in \mathscr{A}$ such that $B \subset A$ and $\nu(B) = b$. One has for the above S that a c-left invariant mean has the intermediate property on any $A \subset S$ for which $cA \subset A$. Moreover, if A is c-left almost convergent, there is even a set $B \subset A$ which is c-left almost convergent for which $\phi(B) = b$.

As a consequence of the above corollary, one has that, if S is a right cancellation left amenable semigroup, (and in particular an amenable group) which contains an element of infinite order, then the range of any left invariant mean ϕ is the whole $[0, 1]$ interval and, moreover, $A \subset S$ need only range in the class of left almost convergent subsets of S in order that $\phi(A)$ should fill the entire $[0, 1]$ interval.

We ask now what is the range of a left invariant mean when S is a group which is the full direct product of countably many copies of the multiplicative group $\{-1, 1\}$. This group is infinite and each of its elements has order two. The above theorem does not answer this question. Another question is what is the range of a left invariant mean on the group of all the permutations of the integers which leave all but a finite number of integers fixed. Both these groups are locally finite infinite groups. We shall show that if G is any amenable group which contains a locally finite infinite group, then the range of any left invariant mean ϕ on $m(G)$ is the whole [0, 1] interval. We show even more than that in Theorem 2.

Notation. G will denote a discrete group with identity e. If $H \subset G$ is a subgroup of G then G/H will always stand for the set of *right* cosets of G with respect to H, i.e., for $\{Hg; g \in G\}$. We say then that $A = \{g_\alpha; \alpha \in I\} \subset G$ is a set of representatives of G/H if $G = HA$ and $Hg_\alpha \cap Hg_\beta \neq \varnothing$ implies $g_\alpha = g_\beta$. We do not speak at all in what follows about left cosets. If $A_1, \ldots, A_n$ are subsets of G, then

$$A_1 \cdots A_n = \{a_1 a_2 \cdots a_n; a_i \in A_i, i = 1, 2, \ldots, n\}.$$

REMARK 1. Let $H \subset H_1$ be subgroups of the group H_2 and let A_1, A_2 be sets of representatives of H_1/H, H_2/H_1, respectively. Then A_1A_2 is a set of representatives of H_2/H. Since $H_1 = HA_1$ and $H_2 = H_1A_2 = H(A_1A_2)$ and if $Ha_1a_2 \cap Ha_1'a_2' \neq \varnothing$ with $a_i, a_i' \in A_i$, $i = 1, 2$, then $H_1a_2 \cap H_1a_2' \neq \varnothing$; hence $a_2 = a_2'$ which implies that $Ha_1 \cap Ha_1' \neq \varnothing$. Therefore, $a_1 = a_1'$. In particular, $a_1a_2 = a_1'a_2'$ with $a_i, a_i' \in A_i$, $i = 1, 2$, is possible if and only if $a_1 = a_1'$ and $a_2 = a_2'$.

LEMMA 1. *Let S_n be subgroups of G such that $S_n \subset S_{n+1}$ for $n = 1, 2, \ldots$. Let R_n be a set of representatives of S_n/S_{n-1} with $e \in R_n$ for each n and $S = \bigcup_{n=1}^\infty S_n$. Then $U_j = \bigcup_{n=j+1}^\infty \{R_{j+1} \cdots R_n\}$ is a set of representatives of S/S_j for $j = 1, 2, \ldots$.*

Proof. $\{R_{j+1} \cdots R_n\}$ is a set of representatives of S_n/S_j for $n \geqq j+1$. Since, by definition, R_{j+1} is a set of representatives of S_{j+1}/S_j and if $\{R_{j+1} \cdots R_{n-1}\}$ is a set of representatives of S_{n-1}/S_j then, by Remark 1, $\{R_{j+1} \cdots R_{n-1}R_n\}$ is a set of representatives of S_n/S_j. Furthermore,

$$S = \bigcup_{n=j+1}^\infty S_n = \bigcup_{n=j+1}^\infty \{S_jR_{j+1} \cdots R_n\} = S_j\left[\bigcup_{n=j+1} \{R_{j+1} \cdots R_n\}\right] = S_jU_j.$$

Assume now that $S_jr \cap S_jr' \neq \varnothing$ for $r \in R_{j+1} \cdots R_n$, $r' \in R_{j+1} \cdots R_m$. (We can assume that $n \geqq m$.) Since $e \in R_i$ for each i, $R_{j+1} \cdots R_k \subset R_{j+1} \cdots R_kR_{k+1}$. Hence, $r, r' \in R_{j+1} \cdots R_n$ which is a set of representatives S_n/S_j. Therefore, $r = r'$ which finishes this proof.

REMARK 2. In the notation of Lemma 1, let R be a set of representatives of G/S and define

$$V_j = \bigcup_{n=j+1}^\infty \{R_{j+1} \cdots R_n\}R = U_jR.$$

Then V_j is a set of representatives of G/S_j (apply Remark 1). Furthermore, from the definition of V_j, it follows immediately that $R_jV_j = V_{j-1}$ for $j \geqq 2$ (which is a set of representatives of G/S_{j-1}).

REMARK 3. If H is a subgroup of G and V a set of representatives of G/H, then clearly $HV=G$ and $hV \cap h'V \neq \varnothing$, with $h, h' \in H$, implies that $h=h'$. Since if $hv=h'v'$ with $v, v' \in V$, then $Hv \cap Hv' \neq \varnothing$, i.e., $v=v'$ and so $h=h'$.

LEMMA 2. *The above chosen subsets $V_1, V_2, \ldots$ of G satisfy*:

(1) $V_n \supset V_{n+1}$ *for each* $n \geqq 1$.

(2) $S_j V_j = G$ *and* $sV_j \cap s'V_j \neq \varnothing$ *with* $s, s' \in S_j$ *implies* $s=s'$.

(3) $R_j V_j = V_{j-1}$ *for* $j \geqq 2$.

Proof. (1) and (2) are clear from Remarks 2 and 3 while $R_j V_j = V_{j-1}$ by definition.

LEMMA 3. *Let S_n be finite subgroups of G of order $p_n < \infty$ such that $S_n \subset S_{n+1}$ (proper inclusion) for $n=1, 2, \ldots$. Let $S=\bigcup_1^\infty S_n$, $Q_n=\{k/p_n; k=0, 1, \ldots, p_n\}$, and $Q=\bigcup_1^\infty Q_n$. Then there exists a family of subsets of S, $\{A(r); r \in Q\}$ such that*:

(1) $A(0)=\varnothing$, $A(1)=G$.

(2) *If* $r_1 < r_2$ *with* $r_1, r_2 \in Q$, *then* $A(r_1) \subset A(r_2)$.

(3) $\phi(A(r))=r$ *for each* $r \in Q$ *and each S-left invariant mean* $\phi \in m(G)^*$.

Proof. Choose for each n a fixed set R_n of representatives of S_n/S_{n-1} with $e \in R_n$. Since S_n is a proper subgroup of S_{n+1}, we have that $p_n q_n = p_{n+1}$ for some $q_n \geqq 2$ and R_n contains q_{n-1} elements. Furthermore, $Q_n \subset Q_{n+1}$ for each n (since $k/p_n = kq_n/p_{n+1}$). Consider the sets V_n constructed in Lemma 3 with respect to the R_n's. We construct the sets $A(r)$ for $r \in Q_1$ as follows: Assume that

$$S_1 = \{s_1^{(1)}, s_2^{(1)}, \ldots, s_{p_1}^{(1)}\}$$

is an enumeration of the elements of S_1 with $s_1^{(1)}=e$. Let then: $A(0)=\varnothing$, $A(1/p_1)=V_1$, $A(2/p_1)=V_1 \cup s_2^{(1)}V_1, \ldots,$ $A(k/p_1)=V_1 \cup s_2^{(1)}V_1 \cup \cdots \cup s_k^{(1)}V_1$ for $k=1, 2, \ldots, p_1$.

Since $S_1 V_1 = G$ and $sV_1 \cap s'V_1 \neq \varnothing$ if $s \neq s'$ with $s, s' \in S_1$ we have that $\phi(V_1) =1/p_1$ and $\phi(A(k/p_1))=k/p_1$ if $k=0, 1, \ldots, p_1$ for each S-left invariant mean ϕ. Also, $A(r_1) \subset A(r_2)$ if $r_1 < r_2$ and $r_1, r_2 \in Q_1$.

Assume now that sets $A(r)$ have been built for $r \in Q_n$ such that $A(1/p_n)=V_n$, $A(0)=\varnothing$, and such that for some enumeration of the elements of S_n, say $S_n =\{s_1^{(n)}, \ldots, s_{p_n}^{(n)}\}$ with $s_1^{(n)}=e$, $A(k/p_n)=\bigcup_{i=1}^k s_i^{(n)}V_n$ if $1 \leqq k \leqq p_n$. We define then sets $A(r)$, $r \in Q_{n+1}$, as follows: If $k=iq_n$ for $i=0, 1, \ldots, p_n$, then $k/p_{n+1}=i/p_n$. (We define in this case $A(k/p_{n+1})=A(i/p_n)$ where $A(i/p_n)$ is given by the induction hypothesis.) We define $A(1/p_{n+1})=V_{n+1}$ and if $R_{n+1}=\{t_1, \ldots, t_{q_n}\}$ with $t_1=e$, define

$$A\left(\frac{k}{p_{n+1}}\right) = \bigcup_{i=1}^{k} t_i V_{n+1} \qquad \text{for } 1 \leqq k \leqq q_n.$$

(Since $R_{n+1}V_{n+1}=V_n$, we have that $A(q_n/p_{n+1})=A(1/p_n)$ in agreement with the previous definition of $A(1/p_n)$.) We define:

$$A\left(\frac{k}{p_n}+\frac{j}{p_{n+1}}\right) = A\left(\frac{k}{p_n}\right) \cup s_{k+1}^{(n)} A\left(\frac{j}{p_{n+1}}\right)$$
$$\text{for } 1 \leqq j \leqq q_n - 1 \quad \text{and} \quad k = 1, 2, \ldots, p_n - 1.$$

We have

$$(*)\quad \begin{aligned} A\left(\frac{k}{p_n}+\frac{j}{p_{n+1}}\right) &= \left[\bigcup_{i=1}^{k} s_i^{(n)}V_n\right] \cup [s_{k+1}^{(n)}[t_1V_{n+1}\cup\cdots\cup t_jV_{n+1}]] \\ &= \left[\bigcup_{i=1}^{k}\bigcup_{m=1}^{q_n} s_i^{(n)}t_mV_{n+1}\right] \cup [s_{k+1}^{(n)}[t_1V_{n+1}\cup\cdots\cup t_jV_{n+1}]] \end{aligned}$$

if $1\leqq j\leqq q_n-1$ and $1\leqq k\leqq p_n-1$.

Enumerate now the elements of S_{n+1} by:

$$s_1^{(n+1)} = t_1 = e,\ s_2^{(n+1)} = t_2,\ldots,\ s_{q_n}^{(n+1)} = t_{q_n},\ s_{iq_n}^{(n+1)} = s_i^{(n)}t_{q_n}$$

for $i=1, 2,\ldots, p_n$, and

$$s_{iq_n+j}^{(n+1)} = s_{i+1}^{(n)}t_j \qquad \text{if } 1 \leqq j \leqq q_n-1 \quad\text{and}\quad 0 \leqq i \leqq p_n-1.$$

It follows from (*), upon a moment's reflection, that

$$\begin{aligned} A\left(\frac{kq_n+j}{p_{n+1}}\right) &= A\left(\frac{k}{p_n}+\frac{j}{p_{n+1}}\right) = \left[\bigcup_{i=1}^{kq_n} s_i^{(n+1)}V_{n+1}\right] \cup s_{kq_n+1}^{(n+1)}V_{n+1}\cup\cdots\cup s_{kq_n+j}^{(n+1)}V_{n+1} \\ &= \bigcup_{i=1}^{kq_n+j} s_i^{(n+1)}V_{n+1} \qquad \text{if } 0 \leqq k \leqq p_n-1 \quad\text{and}\quad 1 \leqq j \leqq q_n-1. \end{aligned}$$

Furthermore,

$$\begin{aligned} A\left(\frac{kq_n}{p_{n+1}}\right) &= A\left(\frac{k}{p_n}\right) = \bigcup_{i=1}^{k} s_i^{(n)}V_n = \bigcup_{i=1}^{k}\bigcup_{j=1}^{q_n} s_i^{(n)}t_jV_{n+1} \\ &= \bigcup_{m=1}^{kq_n} s_m^{(n+1)}V_{n+1} \qquad \text{for } k = 1, 2,\ldots, p_n. \end{aligned}$$

We have thus constructed sets $A(r)$, $r\in Q_{n+1}$, which coincide with the sets $A(r)$ given by the induction hypothesis for $r\in Q_n$, such that $A(1/p_{n+1})=V_{n+1}$ and for some enumeration of $S_{n+1}=\{s_1^{(n+1)},\ldots,s_{p_{n+1}}^{(n+1)}\}$, with $s_1^{(n+1)}=e$, we have that

$$A\left(\frac{k}{p_{n+1}}\right) = \bigcup_{i=1}^{k} s_i^{(n+1)}V_{n+1} \qquad \text{for } 1 \leqq k \leqq p_{n+1}.$$

We can hence assume that there exists a class of subsets of G, $\{A(r); r\in Q\}$ such that $A(0)=\varnothing$, $A(1)=G$ and such that for each n there is an enumeration of S_n, $S_n=\{s_1^{(n)},\ldots,s_{p_n}^{(n)}\}$ with $e=s_1^{(n)}$ such that $A(k/p_n)=\bigcup_{i=1}^{k} s_i^{(n)}V_n$ for $1\leqq k\leqq p_n$ and $A(0)=\varnothing$. (The delicate point here is that $A(r)$ does not depend on the particular representation of r as k/p_n.)

If $r, r'\in Q$ with $r<r'$, say, then $r, r'\in Q_n$ for some n. Hence, $r=k/p_n<k'/p_n=r'$ and so $A(r)=\bigcup_1^k s_i^{(n)}V_n\subset\bigcup_1^{k'} s_i^{(n)}V_n=A(r')$. Furthermore, by Lemma 2, $sV_n\cap s'V_n=0$ if $s\neq s'$ and $s, s'\in S_n$. Hence, $\phi[A(k/p_n)]=k\phi(V_n)$ for any S-left invariant mean ϕ. But

$$\phi(G) = \phi\left[\bigcup_1^{p_n} s_i^{(n)}V_n\right] = p_n\phi(V_n);\ \text{i.e., } \phi(V_n) = \frac{1}{p_n}.$$

This implies that $\phi(A(k/p_n))=k/p_n$ or that $\phi(A(r))=r$ for each $r\in Q$ and any S-left invariant mean ϕ on $m(G)$. This finishes the proof.

THEOREM 2. *Let S_n be finite subgroups of G with $S_n \subset S_{n+1}$ (proper inclusion) and let $S=\bigcup_{n=1}^{\infty} S_n$. Then there exists a class of subsets of G, $\{A(x); 0 \leqq x \leqq 1\}$ such that*

(1) $A(0)=\varnothing$, $A(1)=G$.

(2) *If* $0 \leqq x < x' \leqq 1$, *then* $A(x) \subset A(x')$.

(3) $\phi(A(x))=x$ *for each* $0 \leqq x \leqq 1$ *and each S-left invariant mean ϕ on $m(G)$.*

Proof. Let p_n be the order of S_n and consider the sets Q_n and $Q=\bigcup_1^{\infty} Q_n$ of Lemma 3. Q is dense in [0, 1]. For $r \notin Q$, choose the sets $A(r)$ constructed in Lemma 3. If $0<x<1$ is such that $x \in Q$, then let $A(x)=\bigcap_{\{r \in Q; r>x\}} A(r)$. Let $0 \leqq x < x' \leqq 1$. If either $x \in Q$ or $x' \in Q$, then clearly $A(x) \subset A(x')$. If x and x' are not in Q, then let $r \in Q$ be such that $x<r<x'$. Then $A(x) \subset A(r) \subset A(x')$. Let now ϕ be any S-left invariant mean on $m(G)$ and $0 \leqq x \leqq 1$ with $x \notin Q$.

If r, $r' \in Q$ are such that $r<x<r'$, then

$$r' = \phi(A(r')) \leqq \phi(A(x)) \leqq \phi(A(r)) = r.$$

Since Q is dense in [0, 1], $\phi(A(x))=x$, which finishes the proof.

REMARK. If G contains an element a of order two and ϕ is a mean whose range is $\{0, 1\}$, then $\frac{1}{2}(L_a\phi+L_{a^2}\phi)$ is an a-left invariant mean whose range is $\{0, \frac{1}{2}, 1\}$.

Let S be a semigroup. S is said to be periodic if for any $s \in S$ there is some $n \geqq 1$ such that s^n is an idempotent (see Ljapin [1, p. 113]). We shall subsequently need the following:

LEMMA 4. *Let S be a right cancellation left amenable semigroup which is periodic. Then S is a group*(1).

Proof. Let E be the set of all idempotents of S. For $e \in E$, let $R_e=\{a \in S; a^r=e$ for some $r \geqq 1\}$. If e_1, $e_2 \in E$ and $a^{r_1}=e_1$ $a^{r_2}=e_2$ for some $a \in S$, then $e_1=a^{r_1 r_2}=e_2$. Hence, $R_{e_1} \cap R_{e_2}=0$ if $e_1 \neq e_2$ and $R_e \cap E=\{e\}$ for each $e \in E$. (For this argument, see Ljapin [1, p. 114].)

R_e contains a right ideal for each $e \in E$. In fact, if $a \in S$, then $(ea)^n$ is an idempotent for some $n \geqq 1$. Hence $(ea)^{2n}=e(ea)^n$ which implies by the right cancellation that $(ea)^n=e$. Therefore, $eS \subset R_e$. Since S is left amenable, any two right ideals have nonvoid intersection and, therefore,

$$R_{e_1} \cap R_{e_2} \neq \varnothing \quad \text{for} \quad e_1, e_2 \in E.$$

This implies that S contains a unique idempotent and, since the right cancellation is present, we get by Ljapin [1, p. 113] that S is a group.

REMARK 4. In particular, any finite right cancellation left amenable semigroup is a group (which is well known).

LEMMA 5. *Let S be a left amenable semigroup for which S' is finite and contains $0<n<\infty$ elements (necessarily S' is a finite group). Then there is a left invariant mean*

(1) Here as in Lemma 5 the condition that S is left amenable can be relaxed to: any two right ideals of S have nonvoid intersection.

ϕ on $m(S)$ for which $\phi(A)$ takes the only values $\{k/n;\ k=0, 1, \ldots, n\}$, when A ranges over all subsets of S.

Proof. Let $S'=\{s_1', \ldots, s_n'\}$ where $s_i \in S$ is a fixed representative of s_i', $1 \leqq i \leqq n$ (till the end of this proof). The semigroup S' is left amenable as a homomorphic image of S (Day [2, p. 515(c)]) and has right cancellation (if $a'c'=b'c'$, then $(ac)'=(bc)'$, i.e., $a(cs)=b(cs)$ for some $s \in S$; hence, $a'=b'$). By Remark 4, S' is a finite group. Assume that s_1' is the identity of S' (otherwise, renumber the s_i's) and let $S_i=\{s \in S;\ s'=s_i'\}$ for $i=1, 2, \ldots, n$. S_1 is a subsemigroup of S since if $a, b \in S_1$, then $(ab)'=s_1's_1'=s_1'$ and so $ab \in S_1$. Furthermore, if $a, b \in S_i$ for some i, then $ac=bc$ for some $c \in S$. If $c \in S_j$, let $s_k'=(s_j')^{-1}$. Then $(cs_k)'=s_j's_k'=s_1'$; hence, $a(cs_k)=b(cs_k)$ and $cs_k \in S_1$. We have shown that if $a, b \in S_i$, there is some d in S_1 such that $ad=bd$. In particular, this is true for $a, b \in S_1$. Hence, by Corollary 3 of T. Mitchell [5] (for a different proof, see Theorem 3 in [4]), $m(S_1)$ admits a left invariant mean ϕ which is multiplicative on $m(S_1)$ (i.e., $\psi(fg)=\psi(f)\psi(g)$ for $f, g \in m(S_1)$). Let $\beta(S) \subset m(S)^*$ be the set of multiplicative means on $m(S)$ (i.e., which satisfy $\phi(fg)=\phi(f)\phi(g)$ for $f, g \in m(S)$). Then $L_s(\beta(S)) \subset \beta(S)$ for any s in S and, in particular, for any s in S_1. Since $\beta(S)$ is compact hausdorff there exists by Mitchell's fixed point theorem (see T. Mitchell [5] or [4, Theorem 3]) some $\phi_0 \in \beta(S)$ which is S_1-left invariant. If $A \subset S$, then $1_A^2=1_A$ and, therefore, $\phi_0(A)$ is either 0 or 1 for any $A \subset S$. If $a \in S_i$, then $ac=s_ic$ for some $c \in S_1$ and, therefore:

$$L_a\phi_0 = L_aL_c\phi_0 = L_{ac}\phi_0 = L_{s_ic}\phi_0 = L_{s_i}\phi_0,$$

since ϕ_0 is S_1-left invariant. Furthermore, for any $a \in S$,

$$\{(as_i)';\ i = 1, 2, \ldots, n\} = \{a's_i',\ i = 1, 2, \ldots, n\} = \{s_i',\ i = 1, 2, \ldots, n\} = S',$$

since S' is a group. Let

$$\phi = \frac{1}{n}\sum_1^n L_{s_i}\phi_0.$$

Then

$$L_a\phi = \frac{1}{n}\sum_1^n L_{as_i}\phi_0 = \frac{1}{n}\sum_1^n L_{s_i}\phi_0$$

which shows that ϕ is a left invariant mean. If $A \subset S$, then $(L_{s_i}\phi_0)(1_A)$ is either 0 or 1 since $L_{s_i}\phi_0$ is also multiplicative; therefore, $\phi(A)=(1/n)\sum_1^n L_{s_i}\phi_0(1_A)$ takes one of the values $\{k/n;\ k=0, 1, \ldots, n\}$. Fix now $1 \leqq i, j \leqq n$ and let $s_k'=s_i'(s_j')^{-1}$. If $a \in S_j$, then $(s_ka)'=s_k's_j'=s_i'$ which shows that $s_kS_j \subset S_i$. Therefore, for *any* left invariant ϕ_1 on $m(S)$, $\phi_1(S_j) \leqq \phi_1(s_kS_j) \leqq \phi_1(S_i)$. By symmetry, one has $\phi_1(S_i)=\phi_1(S_j)$. Since S_i are disjoint, one has $1=\phi_1(S)=n\phi_1(S_i)$ or $\phi_1(S_i)=1/n$ for $i=1, 2, \ldots, n$. Therefore, if $A_k=\bigcup_1^k S_i$ and $A_0=\varnothing$ then $\phi_1(A_k)=k/n$ for any left invariant mean ϕ_1 on $m(S)$ and, in particular, the range of the above defined left invariant mean ϕ is exactly the set $\{k/n;\ k=0, 1, \ldots, n\}$. This range is attained on the class of left almost convergent subsets of S.

REMARK 5. For each $0<n<\infty$ there is a countable commutative (left) amenable semigroup S such that S' has order n and each (left) invariant mean on $m(S)$ annihilates single point sets.

Let $T=\{1, 2, \dots\}$ with the multiplication $ij=\max\{i, j\}$ and let G be the cyclic group of order n. Then $S=T\times G$ is (left) amenable since it is commutative. Assume that $(t_0, g_0)\in S$ is such that $0<d=\psi\{(t_0, g_0)\}$ for some left invariant mean ψ on $m(S)$ and let e be the identity of G. Since $(t, g_0^{-1})(t_0, g_0)=(t, e)$ if $t\geqq t_0$, we get that $\psi\{(t_0+k, e)\}\geqq d$ for $k=1, 2, \dots$ which implies that

$$1 \geqq \psi\{(t_0+1, e), \dots, (t_0+k, e)\} \geqq kd$$

for any k. Hence, $d=0$ and any left invariant mean on S annihilates single point sets. It remains to be shown that the order of S' is n. We show that $a'=b'$ if and only if $a, b\in(T, g)$ for some $g\in G$. If $a=(t_1, g)$, $b=(t_2, g)$ and $t=\max(t_1, t_2)$, then $(t_1, g)(t, e)=(t, g)=(t_2, g)(t, e)$. Hence, $a'=b'$. Conversely, if $a'=b'$ and $a=(t_1, g_1)$, $b=(t_2, g_2)$ then for some $(t, g)\in S$, $(t_1, g_1)(t, g)=(t_2, g_2)(t, g)$. Hence, $g_1g=g_2g$ or $g_1=g_2$. Therefore, $a, b\in(T, g_1)$. Thus, $\{(T, g); g\in G\}$ are the different equivalence classes which give rise to the different elements of S'. It can be readily checked that S' is isomorphic to G in this case.

LEMMA 6. *Let $\rho: S\to T$ be a homomorphism of the left amenable semigroup S onto the semigroup T. Assume that there is a family of subsets of T, $\{B(x); 0\leqq x\leqq 1\}$ which satisfies*:

(1) *$B(0)=\varnothing$, $B(1)=T$ and $B(x)\subset B(x')$ if $0\leqq x<x'\leqq 1$.*

(2) *$\psi(B(x))=x$ for each $0\leqq x\leqq 1$ and any left invariant mean ψ on $m(T)$.*

Then $\{A(x)=\rho^{-1}[B(x)], 0\leqq x\leqq 1\}$ satisfies

(1)′ *$A(0)=\varnothing$, $A(1)=S$ and $A(x)\subset A(x')$ if $0\leqq x<x'\leqq 1$.*

(2)′ *$\phi[A(x)]=x$ for any left invariant mean ϕ on $m(S)$ and each $0\leqq x\leqq 1$.*

Proof. It is clear that $A(x)$ satisfy (1)′.

Define $F: m(T)\to m(S)$ by $(Ff)(s)=f(\rho(s))$. Let $\phi\in m(S)^*$ be a left invariant mean. Then $F^*\phi\in m(T)^*$ defined by $(F^*\phi)f=\phi(Ff)$ is a left invariant mean on $m(T)$ (see Day [2, p. 515(c) and pp. 531–532]). But $(F1_{B(x)})(s)=1_{B(x)}(\rho(s))=1_{A(x)}(s)$ for each s in S. Therefore,

$$x = (F^*\phi)[1_{B(x)}] = \phi[1_{A(x)}] = \phi[A(x)]$$

for each $0\leqq x\leqq 1$.

We prove now the main theorem of this paper.

THEOREM 3. *Let S be a left amenable semigroup for which S' is not an AB group. (S' is necessarily a left amenable right cancellation semigroup which is a group in case it is periodic.)*

(1) *If S' is infinite, then the range of each of its invariant means is the whole $[0, 1]$ interval. Moreover, there are, in this case, sets $A(x)\subset S$ such that $A(0)=\varnothing$, $A(1)=S$, $A(x)\subset A(x')$ if $0\leqq x<x'\leqq 1$, and $\phi(A(x))=x$ for each left invariant mean ϕ on $m(S)$. (In particular, ϕ attains its range on left almost convergent sets.)*

(2) *If the range of each left invariant mean on* $m(S)$ *is the whole* $[0, 1]$ *interval, then* S' *is infinite.*

Proof. We first prove (2). If S' contains $0<n<\infty$ elements, then by Lemma 5 there is a left invariant mean ϕ on $m(S)$ whose range is the set $\{k/n; k=0, 1, \ldots, n\}$. (There are even $A_i \subset S$ with $\varnothing = A_0$, $A_n = S$, $A_i \subset A_{i+1}$ for $0 \leqq i \leqq n-1$ and such that $\phi(A_k) = k/n$ for $k=0, 1, \ldots, n$ and each left invariant mean ϕ on $m(S)$.)

We prove now (1):

(a) If S' contains an element of infinite order, then apply the corollary to Theorem 1 and Lemma 6.

(b) If S' does not contain elements of infinite order, then S' is periodic which implies by Lemma 5 that S' is a group which by our assumption contains an infinite locally finite subgroup $T_0' \subset S'$. There are now finite subgroups $S_n' \subset T_0'$ with $S_n' \subsetneqq S_{n+1}'$, $n=1, 2, \ldots$. Let $S_0' = \bigcup_1^\infty S_n'$. As a locally finite group, T_0' is amenable (Day [2, p. 517(k')]). Apply now Theorem 2 (with $G=S'$ and $S=S_0'$) and Lemma 6.

Consequence.

(a) If S is a semigroup which admits a left invariant mean ϕ for which $\phi(A) \neq (1/\sqrt{2})$ (say) for each $A \subset S$ (or even for each left almost convergent $A \subset S$), then S' is either an AB group or a finite group.

(b) If S' is not an AB group, then the set $\{\phi(A); A$ ranges over all left almost convergent subsets of $S\}$ is either the entire $[0, 1]$ interval for each left invariant mean $\phi \in m(S)^*$, or the set $\{k/n; k=0, 1, \ldots, n\}$, for some $0<n<\infty$, for each left invariant mean $\phi \in m(S)^*$.

(c) Is there a finitely additive translation invariant measure ϕ on the set of all subsets of the additive integers Z such that $\phi(Z)=1$ and $\phi(A)$ is rational, say, for any $A \subset Z$? Any such ϕ is, as is easily checked, the restriction of some invariant mean ψ on $m(Z)$, to $\{1_A; A \subset Z\}$. Theorem 3 implies hence that the answer to this (and in fact to a much more general) question is negative.

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